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Calculation of turbulent boundary layers with transpiration and pressure gradient effects

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Abstract—The integral form of the momentum equation is solved by use of a combined law of the wall and wake for the velocity profile appropriate to a transpired turbulent boundary layer. This gives a nonlinear ordinary differential equation which is solved numerically to predict the skin friction coefficient, C_f, variation with distance, as well as integral thicknesses, for both constant blowing fraction, F, and for F variable with x along the surface. Predicted skin friction coefficients are compared to experimental data for both blowing and suction in both zero and nonzero pressure gradients. Predictions compare well with the experimental values.

INTRODUCTION

Prediction of the hydrodynamics characteristics of a turbulent boundary layer flow, with the combined influences of blowing or suction and pressure gradient, is a problem of interest and importance to the aerodynamics engineer. This yields the skin friction coefficient variation along the surface needed for calculating viscous drag on the body. In addition, the solution to the hydrodynamic field is a necessary precursor to the solution of the associated convective heat transfer problem.

Cousteix, in an article contained in Vol. II of the proceedings of the second Stanford Conference, Kline *et al.* [1], cites simplicity, speed and accuracy of integral methods as reasons for them being an important part of the spectrum of tools, along with finite difference methods, for solution of boundary layer problems. Das and White [2] and Das [3] employed integral methods using the combined law of wall and wake to solve for the skin friction distribution in nontranspired boundary layers. Torii *et al.* [4] developed a model based on assumptions asserting that certain functions were the same in nontranspired and transpired turbulent boundary layers, along with an approximation to relate the momentum thickness Reynolds number to the length Reynolds number, to solve for skin friction. Thomas and Kadry [5] express the shear stress variation across the boundary layer as an approximating sequence of polynomials. Then, using mixing length expressions, numerical integration gives a numerical velocity profile. This numerical velocity profile serves as input to the integral x momentum equation which is then solved by finite differences for the skin friction distribution. Predictions are given for a range of constant values of blowing fraction for a zero pressure gradient and an adverse pressure gradient. There is some relatively complicated numerical work involved in using their integral method. Kline *et al.* [1], gives descriptions of, and results for, many finite difference approaches to the solution of the governing partial differential equations of the turbulent transpired boundary layer. Included among these are low Reynolds number k–ε models and other two equation models of turbulence.

In the present work, the velocity profile needed in the integral x momentum equation for turbulent transpired flow was constructed by combining the inner law form suggested by Stevenson [6] with an outer, or wake law form developed by Silva-Freire [7] using asymptotic analysis. The result is a combined law of wall and wake whose overall form is simpler than the well known form of the combined law due to Coles [8]. The integrations needed in the x momentum equation are all performed analytically and this leads to a nonlinear ordinary differential equation which is solved by a standard Runge–Kutta procedure. This solution yields the distribution of various integral thicknesses and the local skin friction coefficient. Unlike previous work, the present procedure is also

NOMENCLATURE

a, b	constants in expression for free stream velocity or blowing fraction
B	defined by equation (13)
B_0, B_1, B_2, B_a, B_b	defined by equations in the Appendix
C	defined by equations (2), (16) and (17)
C_f	$2\tau_w/\rho u_s^2$
F	v_w/u_s blowing fraction
k	turbulent kinetic energy
K	0.41 von Karman's constant
L	reference lengths
P	static pressure
Re_x, Re_θ	$u_s x/\nu$, $u_s\theta/\nu$ Reynolds numbers
u, u_s	local x component and freestream velocity
u^*	$\sqrt{(\tau_w/\rho)}$ friction velocity
u^+, u_s^+	u/u^*, u_s/u^*
v_w	local y component of velocity at the surface
v_w^+	v_w/u^*
x	space coordinate along the surface
y	space coordinate perpendicular to the surface
y^+	yu^*/ν.

Greek symbols

β	defined in equation (21); Clauser's equilibrium parameter
Γ	defined in equation (7)
δ	local hydrodynamic boundary layer thickness
δ^+	$\delta\, u^*/\nu$
δ^*	local displacement thickness
Δ	defined in equation (8)
ε	turbulent dissipation
θ	local momentum thickness
ν	kinematic viscosity
π	Coles wake strength
$\tilde{\pi}$	transpiration wake parameter of Silva-Freire
ρ	mass density
σ	defined by equation (4)
τ_w	local wall shear stress.

applied to situations where F is variable and also where the simultaneous effects of variable F and variable pressure with x are present.

ANALYSIS

Consider steady on the average, two dimensional planar, constant property, turbulent boundary layer flow over a porous surface through which the same fluid is blown into, or extracted from the boundary layer. With the use of inner variables, $u^+ = u/u^*$, $y^+ = yu^*/\nu$ where $u^* = \sqrt{(\tau_w/\rho)}$, and defining the blowing fraction, F, as $F = v_w/u_s$, the integral form of the x momentum equation can be written:

$$u_s \frac{d}{dx}\int_0^{\delta^+} u^+\,dy^+ - \frac{d}{dx}\int_0^{\delta^+} u^* u^{+^2}\,dy^+ + \frac{u_s}{u^*}\frac{du_s}{dx}\delta^+ = \frac{u^{*2}+u_s^2 F}{\nu}. \quad (1)$$

To solve this equation, a velocity profile is needed, $u(x, y^+)$, and will be taken as the combined inner and outer law, the combined law of the wall and wake, for a turbulent transpired boundary layer. This is also the procedure employed by Das and White [2] for the nontranspired boundary layer. Coles [8] form is available for use. However, a simpler form can be developed from a simpler outer law derived by Silva-Freire [7] using asymptotic expansions. If one takes the outer law deduced by Silva-Freire [7], equation (34) of his paper, and requires it to overlap the bilogarithmic inner law given in Stevenson [6], the result is the combined law of wall and wake given next (A procedure for doing this is outlined in Stevenson [9].):

$$u^+ = \frac{1}{K}\ln y^+ + C + \frac{v_w^+}{4}\left[\frac{1}{K}\ln y^+ + C\right]^2 + \left[\frac{\pi + v_w^+\tilde{\pi}}{K}\right]W\left(\frac{y}{\delta}\right) \quad (2)$$

$W(y/\delta)$ is Coles wake function and is approximated well by a commonly used polynomial form due to Moses (White [10]), namely

$$W\left(\frac{y}{\delta}\right) = 2\left[3\left(\frac{y}{\delta}\right)^2 - 2\left(\frac{y}{\delta}\right)^3\right]. \quad (3)$$

In equation (2), if there is no transpiration, $v_w^+ = 0$ and it reduces to the standard combined law of wall and wake. C is the additive constant in the inner law for nontranspired boundary layers, but is a function of x, $C(x)$, in a transpired flow where it may depend upon v_w^+ or F. $\pi = \pi(x)$ is the wake parameter, or wake strength, of Coles, while $\tilde{\pi}(x)$ is a second wake strength parameter, resulting from Silva-Freire's [7] analysis, and depends upon a transpiration parameter such as v_w^+ or F. More will be said later about the calculation of both $\pi(x)$ and $\tilde{\pi}(x)$.

Define a variable σ, which allows a more compact representation of the velocity profiles, as follows.

$$\sigma = \frac{1}{K}\ln y^+ + C(x). \quad (4)$$

Using σ, Coles [8] form of the velocity profile for transpired turbulent boundary layers is given by,

$$u^+ = \sigma + \frac{v_w^+}{4}\sigma^2 + \frac{\pi}{K}W\left(\frac{y}{\delta}\right)\left\{1 + \frac{v_w^+}{4}\left[2\sigma + \frac{\pi}{K}W\left(\frac{y}{\delta}\right)\right]\right\}. \quad (5)$$

The velocity profile, equation (2), developed using the Silva-Freire [7] outer law can be written as follows.

$$u^+ = \sigma + \frac{v_w^+}{4}\sigma^2 + \frac{\Gamma}{K}W\left(\frac{y}{\delta}\right) \quad (6)$$

$$\Gamma = \pi + v_w^+ \tilde{\pi}. \quad (7)$$

The increased complexity of the Coles form, equation (5), is apparent from the cross product and the square terms involving the wake function, σW and W^2. In addition, in figures 6 and 7 of Silva-Freire [7], the outer law velocity profile implied by equation (6) is compared to the outer law implied by Coles [8] form, equation (5), for a range of blowing and suction parameters, F. Agreement with the experimental velocity profile data is seen to be about the same for both forms. Thus, in order to make the integrations required in equation (1) more simply, it was decided to use equation (6) rather than (5) at this time.

When making the integrations needed in equation (1), it was noticed that they could be done more simply and faster if the integration variable was changed to σ defined in equation (4). Call the value of σ, at the edge of the boundary layer, $y^+ = \delta^+$, Δ, where

$$\Delta = \frac{1}{K}\ln\delta^+ + C(x). \quad (8)$$

Thus one of the needed integrals becomes

$$\int_0^{\delta^+} u^+\,dy^+ = K e^{-KC}\int_{-\infty}^{\Delta} u^+ e^{K\sigma}\,d\sigma \quad (9)$$

$u^+(\sigma)$ is given by equation (6) and $W(\sigma)$, from equation (3) is written as,

$$W(\sigma) = 2\,e^{2K(\sigma-\Delta)}[3 - 2\,e^{K(\sigma-\Delta)}]. \quad (10)$$

Doing this allows both integrations in equation (1) to be performed analytically with the results as follows:

$$\int_0^{\delta^+} u^+\,dy^+ = \delta^+\left[\Delta - \frac{1}{K} + \frac{v_w^+ B_0}{4} + \frac{\Gamma}{K}\right] \quad (11)$$

$$\int_0^{\delta^+} u^{+2}\,dy^+ = \delta^+\left[B + \frac{\Gamma}{K}\left(B_a + \frac{B_b v_w^+}{2} + \frac{52\Gamma}{35K}\right)\right] \quad (12)$$

$$B = B_0 + B_1 v_w^+ + B_2 v_w^{+2}. \quad (13)$$

The expressions for B_0, B_1, etc. are given in the Appendix as a function of Δ.

Inserting equations (11) and (12) into equation (1) gives the result which follows next.

$$E_1\frac{d\delta^+}{dx} + E_2\frac{dF}{dx} + E_3\frac{dC}{dx} + E_4\left[\frac{d\pi}{dx} + v_w^+\frac{d\tilde{\pi}}{dx}\right] = E_5. \quad (14)$$

The coefficients, E_1, E_2, etc. in equation (14) are algebraic functions of δ^+, F, π, $\tilde{\pi}$ and C_f. The blowing fraction, F, is a known prescribed function of x in any given problem. Hence, additional relations are needed for C_f, C, π and $\tilde{\pi}$ before solution of the equation for $\delta^+ = \delta^+(x)$ can be accomplished.

C_f is related to δ^+ and other variables by evaluating the velocity profile, equation (2), at $y^+ = \delta^+$ where $u^+ = u_s^+$, giving, after introducing the blowing fraction $F = v_w/u_s$, the following:

$$u_s^+ = \sqrt{\frac{2}{C_f}} = \frac{\Delta + \dfrac{2\pi}{K}}{1 - \dfrac{F\Delta^2}{4} - \dfrac{2F\tilde{\pi}}{K}}. \quad (15)$$

Expression for C

Various suggestions have been advanced for the form of the function C appearing in equation (2). Stevenson [6] discusses many of these proposed forms. Coles [8] endorses the form recommended by Simpson [11] who chooses C such that equation (2) gives $u^+ = 11$ at $y^+ = 11$ which makes C a specified function of v_w^+. The latest proposal for C is given by Silva-Freire [7] and is based upon blowing data used at the second Stanford Conference, Kline *et al.* [1]. The relation which he developed is shown next.

$$C = 5 - 512F \quad (F > 0). \quad (16)$$

Though based solely on blowing data, he suggests the validity of this result for suction ($F < 0$) also.

An earlier relation specifically for suction is given by Bradshaw [12] as being,

$$C = 5 + 1375 v_w^{+2} \quad (F < 0). \quad (17)$$

Equations (16) and (17) are used in the current work. One case of suction also was solved using equation (16) instead of (17) and it was found that predicted skin friction coefficients, C_f, differed by only 0.4%.

Wake parameter relations

Relations are needed for the wake strength parameter, π, of Coles [8] and the $\tilde{\pi}$ of Silva-Freire [7].

By use of the experimental data of Andersen *et al.* [13], Silva-Freire [7] deduced the following form for his $\tilde{\pi}$.

$$\tilde{\pi} = -1.95\ln|F| - 3.1 \quad (18)$$

thus,

$$\frac{d\tilde{\pi}}{dx} = -\frac{1.95}{F}\frac{dF}{dx}. \quad (19)$$

A correlation of experimental data by White [14] led to an expression for π in terms of the Clauser pressure gradient parameter β. More recently, the inclusion of far greater numbers of experimental data points led to improved relations for π shown and discussed in Das and White [2] and Das [3]. These culminated in the recommendation of the following relation for π given in White [10]:

$$\beta = -0.4+0.76\pi+0.42\pi^2 \tag{20}$$

$$\beta = \frac{\delta^*}{\tau_w}\frac{dP}{dx}. \tag{21}$$

The present authors saw two problems with equation (20). First, it gives $\pi = 0.426$ when $\beta = 0$, yet most flat plate data gives values considerably higher than this. The data of Wieghardt and others, shown in Coles [8], indicate a π value in excess of 0.50 for $\beta = 0$. The β vs π relation for favorable pressure gradients in Das and White [2] gives $\pi = 0.55$ at $\beta = 0$. Secondly, equation (20) indicates vanishing of the wake, $\pi = 0$, at $\beta = -0.4$. There is theoretical evidence, Mellor and Gibson [15], that for equilibrium flows (where β is sensibly constant) the wake vanishes at $\beta = -0.5$. On the basis of these arguments, the present authors replaced -0.4 by the -0.5 in equation (20). This causes the wake to vanish at $\beta = -0.50$ and gives a value of $\pi = 0.5127$ at $\beta = 0$, a value more nearly in accord with experiment for this case. With this change, the β–π relation used in the present work is shown next

$$\beta = -0.5+0.76\pi+0.42\pi^2. \tag{22}$$

Theoretically, a relation such as equation (22), which gives π in terms of β alone, would hold only for equilibrium flows where the Clauser parameter β (equation (21)) is constant. In fact, Das and White [2] show that it tends to correlate nonequilibrium flow data also. Certainly their use of π–β relations to calculate skin friction variation in nonequilibrium flows, Das and White [2], Das [3], yield predictions in good accord with the data. The data that led to equation (22) were for nontranspired flows, but Coles [8] finds that π is insensitive to transpiration to or from the boundary layer.

With the introduction of equations (16) and (17) for C, (18) for $\tilde{\pi}$ and (21) and (22) for π, closure of the problem is achieved with equations (14) and (15). Combining all of these relationships yields the following nonlinear differential equation for δ^+.

$$\frac{d\delta^+}{dx} = G_0 - G_1\frac{dF}{dx} - G_2\frac{d^2u_s}{dx^2} - G_3\frac{du_s}{dx} - G_4\frac{d\tilde{\pi}}{dx}. \tag{23}$$

F, the blowing fraction, is a given, prescribed function of x. $u_s(x)$ is available from the potential flow solution or from measurement and allows calculation of the needed derivatives of u_s. Equation (19) gives $d\tilde{\pi}/dx$. G_0, G_1, etc. are highly complicated known algebraic functions of δ^+, u_s^+, $\tilde{\pi}$, etc.

Next, equation (23) is solved for $\delta^+(x)$ by use of a fourth order Runge–Kutta finite difference procedure. The lattice spacing, Δx, was refined until the solution was effectively independent of the spacing. For example, in a case where $F \sim -0.004x^{-0.17}$ and $u_s \sim x^{-0.15}$, a final spacing of $\Delta x = 0.0145$ m was used. Halving this value produced a maximum change of less than 0.07% in the value of the predicted $C_f/2$. To begin the solution of equation (23), an initial condition is required. Usually what is available is the value of the skin friction coefficient, C_f, at $x = x_0$, not the boundary layer thickness δ^+. The starting value of δ^+ was found by trying a succession of increasing values of δ^+ at x_0 to evaluate the coefficients, G_0, $G_1, \ldots$ and then solve the equation for C_f at $x_0+\Delta x$ where Δx is taken extremely small, say $10^{-6}x_0$. When the solution at $x_0+\Delta x$ yields the known value of C_f at x_0, the initial value of δ^+ is determined. Alternately, one can take equations (8), (15)–(18), (21) and (22) and solve these algebraic relations iteratively to find δ^+ at x_0. A similar procedure is used if one starts with a known value of momentum thickness Reynolds number.

Once $\delta^+(x)$ has been solved, it is used in equation (15) to solve for $C_f(x)$. Integral thicknesses, such as momentum thickness θ and displacement thickness δ^*, and their associated Reynolds numbers can then be solved for from their definitions.

Validity limits of method

The range of conditions for which the present method is expected to be valid depends on the conditions for which the velocity profile, equation (6), and the β–π relation, equation (22), are expected to hold. Coles [8], indicates that the velocity profile, equation (5), would certainly be applicable with the following limits on F, $-0.004 \leqslant F \leqslant 0.010$. It seems reasonable to expect this to be the case for equation (6) as well. The β–π relation being used, equation (22), seems to be sufficient for all pressure gradients except severe adverse gradients leading practically to separation, see White [10]. The present authors, also, have achieved good predictive success using equation (22), for $F = 0.0$, in all but very severe adverse pressure gradients. In fact, Das and White, [2], used two different β–π relations, one for severe adverse pressure gradient and one for separated flow, in their predictions of flows which actually separated. It is also known, Mellor and Gibson [15], that in severely accelerated flows the wake vanishes as $\beta \to -0.5$. Yet, the wake parameter, $\tilde{\pi}$, of Silva-Freire [7] is independent of β. Thus, in such a flow, the present authors recommend setting $\tilde{\pi} = 0$ in the velocity profile, equation (6).

RESULTS AND DISCUSSION

The variation of the skin friction coefficient with position x along the surface is usually the result of

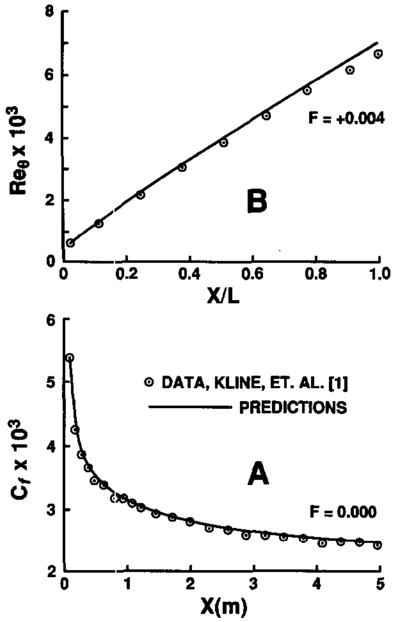

Fig. 1. Predictions compared to data. (A) $F = 0.00$, u_s = constant. (B) $F = +0.004$, u_s = constant. $L = 2.286$ m (7.5 feet).

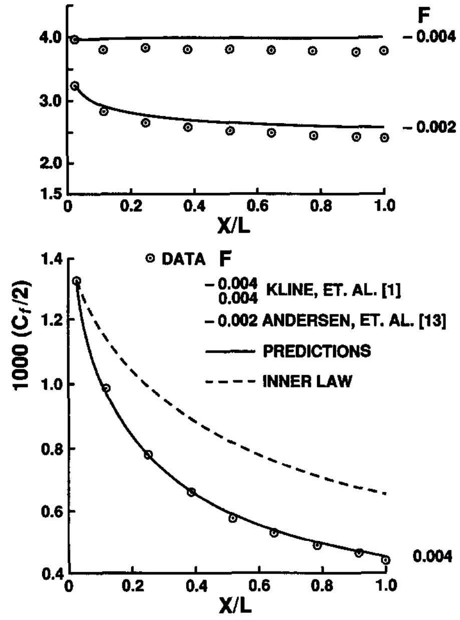

Fig. 2. Predicted skin friction compared to data. $F = +0.004$, u_s = constant; $F = -0.002, -0.004$, $u_s \sim x^{-0.15}$. $L = 2.286$ m (7.5 feet).

most interest to the flow and heat transfer analysts. Hence, the bulk of the predicted results are for $C_f/2$ along with some cases of momentum thickness Reynolds number, Re_θ, and of displacement thickness Reynolds number, Re_δ^*, variation. Predicted values of $C_f/2$ will be compared to experimental data for both blowing, $F > 0$, and suction, $F < 0$, for cases where F is constant with x and cases where F varies with position x along the surface, for both constant free stream velocity, u_s, and variable free stream velocity, particularly decelerating flows.

In the prediction of $C_f/2$ by use of the present method, all calculations begin by matching the experimental value of $C_f/2$ at the first data point, unless noted otherwise. Similarly, predictions of Re_θ and Re_δ^* begin by matching the experimental value of momentum and displacement thickness Reynolds number at the first data point.

Figure 1A shows predicted values of $C_f/2$ as the solid curve compared to experimental data, the circles. This is for the flat plate with zero transpiration, $F = 0$, and constant free stream velocity. The experimental data is from Wieghardt, case 0612 of the Second Stanford Conference, Kline *et al.* [1]. As is seen, agreement of the present predictions with data is very good for this simple baseline test case used as a check on the method.

Figure 1B shows results for strong blowing, $F = +0.004$, with constant free stream velocity u_s. This was one of the transpiration tests, case 024, of the 2nd Stanford Conference. The data is from Andersen *et al.* [13]. This figure shows predicted momentum thickness Reynolds number compared to experimental data. Agreement is considered very good. The maximum difference between predictions and data is about 6% at the last data point. Skin friction coefficient prediction for this case, $F = +0.004$, is shown on the lower half of Fig. 2. The lower, solid, curve is the prediction employing the combined law of the wall and wake, equation (2), and exhibits excellent agreement with data. In fact, if one compares the present predictions for $F = +0.004$ to the numerical finite difference method solutions for this case presented in Vol. III of the 2nd Stanford Conference proceedings [1], the present predictions are better than three of the six methods and are comparable to those of the remaining three methods. These six methods encompassed two equation models of turbulence, such as k–ε models, as well as Reynolds stress models. For comparison purposes, the dashed line in Fig. 2 for $F = +0.004$ is the prediction using the inner law alone, just the Law of the Wall, as the velocity profile. As can be seen, the $C_f/2$ values are much higher than the data because of the lack of a wake in the velocity profile in a situation, namely strong blowing, where it is expected that the wake effect will be accentuated, Julien *et al.* [16].

The upper most curve in Fig. 2 is for strong suction, $F = -0.004$, combined with a deceleration, $u_s(x) \sim (ax+b)^{-0.15}$. This data is also from the 2nd Stanford Conference [1], case 0242. Again, the present method gives good agreement of predicted $C_f/2$ with the data with a maximum difference of about 5%. This is comparable to the agreement of the five finite difference methods which were applied to this case at the 2nd Stanford Conference (Vol. III [1]). The final curve in Fig. 2 is for moderate suction, $F = -0.002$, along with a decelerating free stream, $u_s(x) \sim (ax+b)^{-0.15}$. The data is from Andersen *et al.*

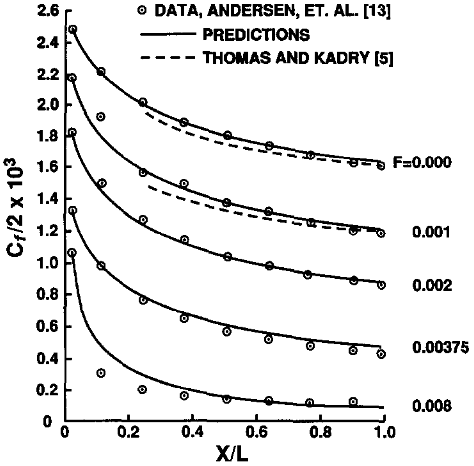

Fig. 3. Predicted skin friction compared to data. u_s = constant. $L = 2.286$ m (7.5 feet).

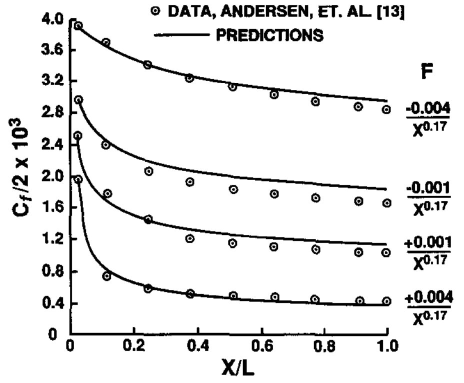

Fig. 4. Predicted skin friction compared to data. $u_s \sim x^{-0.15}$, $L = 2.286$ m (7.5 feet).

[13] and agreement of present predictions with data is good.

The experimental data in Fig. 3 is from Andersen *et al.* [13]. The blowing fraction, F, is constant, ranging from 0.00 to 0.008, and free stream velocity, u_s, is also constant. The solid lines represent the present predictions and agree well with the data except for the case of $F = +0.008$, very strong blowing. Although the data evaluators at the 2nd Stanford Conference [1] endorsed the integrity of most of the data in Andersen *et al.* [13], they indicated that for $F > +0.004$, the boundary layers were not strictly two dimensional and the case of $F = +0.008$, in particular, failed their internal checks. Perhaps this is part of the reason for the quality of our predictions in this case. Also shown in Fig. 3, as dashed lines, are some of the predictions of Thomas and Kadry [5]. As is seen, their predictions, the dashed lines, begin somewhere after the third experimental data point, that is, after the region of the largest gradients in $C_f/2$. Also, it is not known what condition was being used at the start of the calculation. Reference [5] does not indicate how the calculations were started. As was mentioned earlier, the present calculations begin with the experimental value at the first data point as the initial condition.

Data and predictions in Figs. 4 and 5 are presented for the most complicated cases, namely those in which both the free stream velocity, u_s, and the blowing fraction, F, are functions of the distance x along the surface. Again, the experimental data are from Andersen *et al.* [13]. The data in Fig. 4 are for $u_s(x) \sim [a+bx]^{-0.15}$ and $F(x) \sim [a+bx]^{-0.17}$. In general, the agreement between predictions and data in Fig. 4 is satisfactory to good. For the case where $F \sim 0.001x^{-0.17}$, the maximum difference in the $C_f/2$ predictions vs data is about 10% with an average difference of 7%. Predicted values of $C_f/2$ and Re_θ are compared to data in Fig. 5. There is good agreement for the case where $F \sim +0.002x^{-0.17}$ with $u_s \sim x^{-0.15}$. The agreement is not so good for the remaining case where $F \sim -0.002x^{-0.16}$, but where $u_s(x) \sim [a+bx]^{-0.20}$. The dependence of u_s on the -0.20 power of x represents a fairly strong deceleration. It may be that the β–π relation being used, equation (22), has reached its limit of adequacy or that this is happening to the form being used as the wake contribution in equation (2).

In Fig. 6, the data come from Simpson *et al.* [17]. All the data is for a constant value of u_s, but for both constant F and also for F varying as $x^{-0.20}$ and $x^{-0.50}$. As is evident, the agreement of present predictions with data is very good in all four cases. However, in the case where $F \sim -0.0064x^{-0.20}$, we were unable to match the measured value of $C_f/2$ at the first data point. The measured value seemed incompatible with the expressions being used for the velocity profile and for the $\pi(x)$ relation. So, instead the calculation was started by using the lowest value of $C_f/2$ permitted by our expressions. The value of F at the first data point,

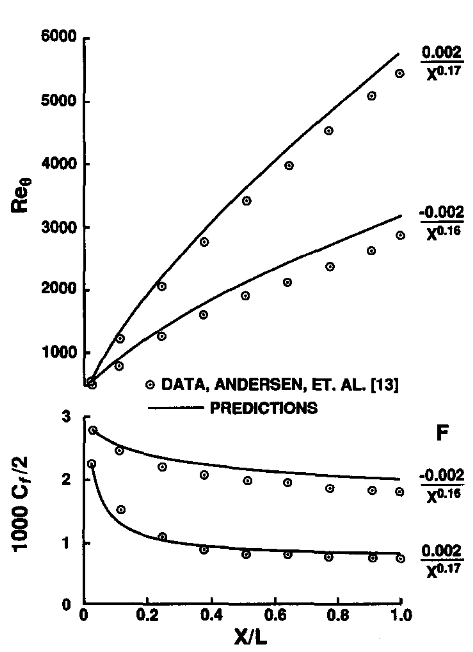

Fig. 5. Predictions compared to data. $F \sim x^{-0.17}$, $u_s \sim x^{-0.15}$, $F \sim x^{-0.16}$, $u_s \sim x^{-0.20}$. $L = 2.286$ m (7.5 feet).

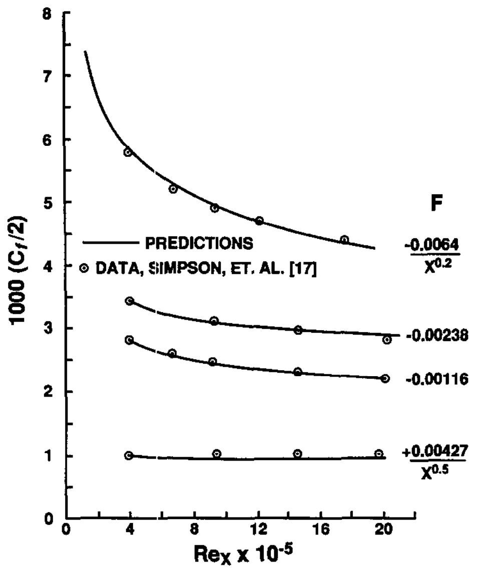

Fig. 6. Predicted skin friction compared to data. u_s = constant.

$F = -0.00726$, seems to force us to use a $C_f/2$ very close to this value, as in the case of the asymptotic suction limit. Possibly the measured value of $C_f/2$ is in error at this point. Simpson *et al.* [17] noted that this case had a "laminar like" velocity profile along with very small measured values of momentum thickness Reynolds number. Perhaps this is the source of our difficulties with the starting point for this one case.

The final figure, Fig. 7, shows our predictions and

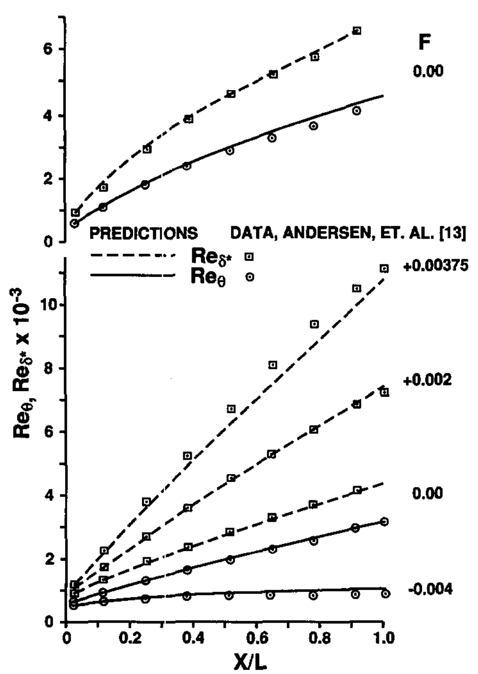

Fig. 7. Predicted momentum and displacement thickness Reynolds numbers compared to data. $F = 0.00$, $u_s \sim x^{-0.20}$; $F = 0.00375$, 0.002 and 0.00, u_s = constant; $F = -0.004$, $u_s \sim x^{-0.15}$. $L = 2.286$ m (7.5 feet).

experimental data from Andersen *et al.* [13] for momentum thickness Reynolds number, Re_θ, and for displacement thickness Reynolds number, Re_δ^*. The dashed lines give the predictions for Re_δ^*, while the solid lines represent the predicted Re_θ. The two curves at the top of the figure, for $F = 0.0$, are for the case of a fairly strong deceleration, $u_s(x) \sim x^{-0.20}$. Agreement of these with the data appears to be good. The predictions and data in the bottom portion of the figure are for the case of u_s being constant, except for the lowest curve, $F = -0.004$, which is for a deceleration, $u_s(x) \sim x^{-0.15}$. The predictions seem to agree well with the data.

CONCLUSIONS

By using inner coordinates, u^+, y^+, the integral form of the x momentum equation has been solved by utilizing the combined law of the wall and wake, in the form appropriate to a transpired turbulent boundary layer, as the needed velocity profile. The predictions of the method compare well to experimental data for the skin friction coefficient for both blowing and suction, constant and variable with x blowing fractions and for both zero and variable pressure gradients. The predictions of the method also compare very favorably with predictions from a number of numerical finite difference methods which solve the partial differential equations governing the time averaged turbulent velocity field.

Information has been provided to allow the construction of a relatively modest computer program to calculate the variables of interest.

It is felt that the basic approach presented here has the potential to be extended to the convective heat transfer problem and this is being worked on at present.

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APPENDIX

Coefficients in equations (11) and (12):

$$B_0 = \Delta^2 - \frac{2\Delta}{K} + \frac{2}{K^2} \tag{A1}$$

$$B_1 = \frac{1}{2}\left[\Delta^3 - \frac{3\Delta^2}{K} + \frac{6\Delta}{K^2} - \frac{6}{K^3}\right] \tag{A2}$$

$$B_2 = \frac{1}{16}\left[\Delta^4 - \frac{4\Delta^3}{K} + \frac{12\Delta^2}{K^2} - \frac{24\Delta}{K^3} + \frac{24}{K^4}\right] \tag{A3}$$

$$B_a = 2\Delta - \frac{5}{6K} \tag{A4}$$

$$B_b = \Delta^2 - \frac{5\Delta}{6K} + \frac{23}{72K^2}. \tag{A5}$$